A recursive algorithmic construction for spherical codes in dimensions $\mathbb{R}^{2^k}$

Henrique K. Miyamoto
Henrique Sá Earp e Sueli Costa

Unicamp - University of Campinas

miyamotohk@gmail.com

July 9, 2018

Introduction

Spherical code

A spherical code $\mathcal{C}(M, n)$ is a set of M points on the surface of the unit Euclidian sphere S^{n-1}:

$$\mathcal{C}(M, n) := \{x_1, ..., x_M\} \subset S^{n-1} \subset \mathbb{R}^n$$

Sphere packing problem

This problem may be presented in two ways:

(i) To distribute on S^{n-1} a given number M of points in a way that maximises their minimum mutual Euclidian distance;

(ii) Given a minimum Euclidian distance $d > 0$, to find the largest possible number M of points on S^{n-1} with all mutual distances at least d.

Construction: basic case

Hopf foliation in $\mathbb{R}^4$

The sphere S^3 is foliated by tori T^2 with parametrisation given by:

$$(\eta, \xi_1, \xi_2) \mapsto (e^{i\xi_1} \sin \eta, e^{i\xi_2} \cos \eta), \quad \eta \in \left[0, \frac{\pi}{2}\right], \; \xi_j \in [0, 2\pi[, \; j = 1, 2$$

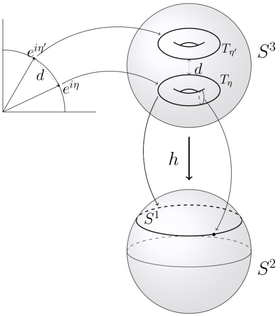

Figure: Hopf foliation and distance between tori in $\mathbb{R}^4$.

Construction: generalisation

Generalisation for $\mathbb{R}^{2n}$: each S^{2n-1} is foliated by $S^{n-1}_{\sin\eta} \times S^{n-1}_{\cos\eta}$.

1 Varying η, choose a family of $S^{n-1}_{\sin\eta} \times S^{n-1}_{\cos\eta}$ distant of d.

2 On each S^{n-1}, do the distribution of the previous dimension up to scaling.

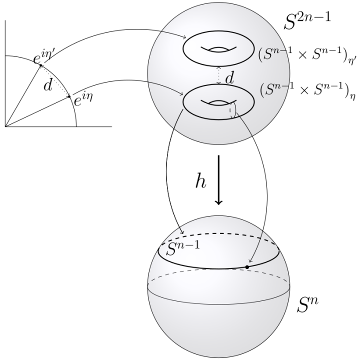

Figure: Hopf foliation and distance between leaves in $\mathbb{R}^{2n}$.

Results

d	SCHF	TLSC	Apple-peeling	Wrapped	Laminated
0.4	280	308	342	*	*
0.2	$2,656$	$2,718$	$2,822$	*	*
0.1	$22,016$	$22,406$	$22,740$	$17,198$	$16,976$
0.01	2.27×10^7	2.27×10^7	1.97×10^7	$2.31 \times 10^{7\dagger}$	2.31×10^7

Table: Comparison with spherical codes in $\mathbb{R}^4$ [Torezzan et al., 2013].

n	d	SCHF	TLSC (k)	TLSC (hyperplanes)	TLSC (polygones)
8	0.9	64	8	8	40
	0.8	144	8	8	128
	0.3	104,512	45,252	61,060	89,945
	0.2	2.28×10^6	3.42×10^5	6.64×10^5	2.15×10^6
16	0.2	6.93×10^{10}	4.76×10^9	7.44×10^9	5.01×10^9
	0.1	4.16×10^{15}	2.41×10^{12}	7.32×10^{12}	2.39×10^{15}
32	0.1	8.66×10^{26}	6.81×10^{21}	1.50×10^{22}	7.02×10^{24}

Table: Comparison with TLSC implementations in $\mathbb{R}^n$ [Naves, 2016].

References

Cristiano Torezzan, Sueli I. R. Costa e Vinay A. Vaishampayan
Constructive spherical codes on layers of flat tori
IEEE Transactions on Information Theory, v. 59, n. 10, p. 6655-6663, out. 2013

David W. Lyons
An elementary introduction to Hopf fibration
Mathematics Magazine, v. 76, n. 2, p. 87-98, apr. 2003

Lígia R. B. Naves
Códigos esféricos em canais grampeados
Thesis (Doctorate in Applied Mathematics) – IMECC, Unicamp, 2016