Non-existence of linear $e$-perfect Lee codes depending on the base-3 representation of $e$

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July 25, 2018 - IMECC, Unicamp
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1. Notations and definitions

2. Perfect Lee codes over $\mathbb{Z}$ - The Golomb-Welch conjecture

3. The Zhang-Ge theorem and our generalization
Basics on Lee codes

Linear codes over \( \mathbb{Z}_q \) and over \( \mathbb{Z} \)
- Let \( R = \mathbb{Z} \) or \( \mathbb{Z}_q \). \( C \) is a code over \( R \) if \( C \subseteq R^n \) for some \( n \in \mathbb{Z}^+ \).
- \( C \) is a linear code if \( C \) is an additive subgroup of \( R^n \).
- A linear code \( C \subseteq \mathbb{Z}^n \) is \( q \)-periodic if \( q\mathbb{Z}^n \subseteq C \).

\[
\left\{ \begin{array}{l}
\text{Linear codes } C \subseteq \mathbb{Z}_q^n \\
\text{Linear codes } C \subseteq \mathbb{Z}^n \\
\end{array} \right\} \leftrightarrow \left\{ \begin{array}{l}
\text{\( q \)-periodic linear codes } C \subseteq \mathbb{Z}_q^n \\
\text{\( q \)-periodic linear codes } C \subseteq \mathbb{Z}^n \\
\end{array} \right\}
\]

The Lee metric

\[
d(x, y) := \begin{cases} 
\min\{|x - y|, q - |x - y|\} & \text{if } x, y \in \mathbb{Z}_q \\
|x - y| & \text{if } x, y \in \mathbb{Z}
\end{cases}
\]
The Lee metric

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\end{cases} \]

Example: The Lee metric over \( \mathbb{Z}_9 \)

Consider \( x = 7, y = 2 \)
The Lee metric

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|x - y| & \text{if } x, y \in \mathbb{Z}
\end{cases} \]

Example: The Lee metric over \( \mathbb{Z}_9 \)

Consider \( x = 7, y = 2 \) \( \Rightarrow |x - y| = 5 \)
The Lee metric

\[ d(x, y) := \begin{cases} 
\min\{|x - y|, q - |x - y|\} & \text{if } x, y \in \mathbb{Z}_q \\
|x - y| & \text{if } x, y \in \mathbb{Z}
\end{cases} \]

Example: The Lee metric over \( \mathbb{Z}_9 \)

Consider \( x = 7, y = 2 \) \( \Rightarrow \) \( |x - y| = 5, q - |x - y| = 4 \)
**The Lee metric**

\[
d(x, y) := \begin{cases} 
\min\{|x - y|, q - |x - y|\} & \text{if } x, y \in \mathbb{Z}_q \\
|x - y| & \text{if } x, y \in \mathbb{Z}
\end{cases}
\]

**Example: The Lee metric over \(\mathbb{Z}_9\)**

Consider \(x = 7, y = 2 \Rightarrow |x - y| = 5, q - |x - y| = 4 \Rightarrow d(7, 2) = 4\).
The Lee metric

\[ d(x, y) := \begin{cases} 
\min\{|x - y|, q - |x - y|\} & \text{if } x, y \in \mathbb{Z}_q \\
|x - y| & \text{if } x, y \in \mathbb{Z} 
\end{cases} \]

- For words \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^n (R = \mathbb{Z}_q \text{ or } \mathbb{Z}) \):
  \[ d(x, y) := \sum_{i=1}^{n} d(x_i, y_i). \]

The Lee balls

For \( x = (x_1, \ldots, x_n) \in R^n (R = \mathbb{Z}_q \text{ or } \mathbb{Z}) \) and \( e \geq 0 \):

\[ B(x, e) := \{ y \in R^n : d(x, y) \leq e \} = x + B(0, e). \]
Lee balls in $\mathbb{Z}_9^2$

Figure. $B((0,0), 0) = \{(0,0)\}$. 
Figure. $B((0,0),1) = \{(0,0), (0, \pm 1), (\pm 1, 0)\}$. 
Lee balls in $\mathbb{Z}_9^2$

Figure. $B((0,0), 2) = \{(0,0), (0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1), (0, \pm 2), (\pm 2, 0)\}$. 
Lee balls in $\mathbb{Z}_9^2$

Figure. $B((7,1),2) = (7,1) + B((0,0),2)$. 
Lee balls in $\mathbb{Z}_9^2$

Figure. $B((2, 5), 2) = (2, 5) + B((0, 0), 2)$. 
Perfect Lee codes

$C \subseteq R^n \ (R = \mathbb{Z}_q \text{ or } \mathbb{Z})$ is an $e$-perfect Lee code if

$$R^n = \bigcup_{c \in C} B(c, e),$$
Perfect Lee codes

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Example of perfect code in $\mathbb{Z}_{13}^2$
Perfect Lee codes

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Example of perfect code in $\mathbb{Z}_{13}^2$
Correspondence between codes over $\mathbb{Z}_q$ and codes over $\mathbb{Z}$

$\{ \text{Linear codes } C \subseteq \mathbb{Z}_q^n \} \leftrightarrow \{ \text{q-periodic linear codes } C \subseteq \mathbb{Z}^n \}$
Correspondence between codes over $\mathbb{Z}_q$ and codes over $\mathbb{Z}$

\[
\begin{align*}
\{ \text{Linear codes } C \subseteq \mathbb{Z}_q^n \} & \quad \Longleftrightarrow \quad \{ \text{$q$-periodic linear codes } C \subseteq \mathbb{Z}^n \} \\
\end{align*}
\]

For $q \geq 2e + 1$:

\[
\begin{align*}
\{ \text{Linear $e$-perfect Lee codes } C \subseteq \mathbb{Z}_q^n \} & \quad \Longleftrightarrow \quad \{ \text{$q$-periodic linear $e$-perfect Lee codes } C \subseteq \mathbb{Z}^n \} \\
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\left\{ \text{$q$-periodic linear $e$-perfect Lee codes} \quad C \subseteq \mathbb{Z}^n \right\}
\end{align*}
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From here on we only consider Perfect Lee codes $C \subseteq \mathbb{Z}^n$. 
1 Notations and definitions

2 Perfect Lee codes over \( \mathbb{Z} \) - The Golomb-Welch conjecture

3 The Zhang-Ge theorem and our generalization
The Golomb-Welch conjecture

PERFECT CODES IN THE LEE METRIC AND THE PACKING OF POLYOMINOES*

SOLOMON W. GOLOMB AND LLOYD R. WELCH†

1. The geometry of Shannon’s five-phase code. In [4] Shannon considered the problem of coding to completely eliminate errors in a channel using a 5-symbol alphabet, with the error pattern as shown in Fig. 1. The alphabet may be regarded as the integers modulo 5. When the integer \( r \) is sent, either \( r \) or \( r + 1 \) is received, with respective probabilities \( p \) and \( q \). If one forms a “code” consisting of sending each symbol \( m \) times to represent the fact that it occurred once in the message, then there is still a probability of \( q^m \) that an error will occur. However, there exists a code using only two code symbols per message symbol which eliminates errors entirely (see Fig. 2). In this code, if \((a, b)\) is a codeword, then it may be received as either \((a, b)\) or \((a + 1, b)\) or \((a, b + 1)\) or \((a + 1, b + 1)\). However, we can associate all four of these received messages uniquely with \((a, b)\) when we use the code of Fig. 2. This is most readily seen via the geometric presentation in Fig. 3. The 25 possible codewords \((a, b)\) are represented by the 25 cells, with coordinates \((a, b)\). The codewords of Fig. 2 correspond to the cells with dots in them. Each dot is

Perfect Lee codes exist for the following parameters:

- For every \( e \geq 1 \) there is an \( e \)-perfect Lee code \( C \subseteq \mathbb{Z}^1 \).
- For every \( e \geq 1 \) there is an \( e \)-perfect Lee code \( C \subseteq \mathbb{Z}^2 \).
- For every \( n \geq 1 \) there is a 1-perfect Lee code \( C \subseteq \mathbb{Z}^n \).

Conjecture: For \( n \geq 3 \) and \( e \geq 2 \), there are no \( e \)-perfect Lee codes in \( \mathbb{Z}^n \).
Some results towards the Golomb-Welch conjecture (Fixed dimension):

- **(Golomb-Welch 1970)**: For $n \geq 3$ fixed. There is an integer $e_n \geq 2$ ($e_n$ unspecified) such that there are no $e$-perfect Lee codes in $\mathbb{Z}^n$ for $e \geq e_n$.

- **(Golomb-Welch 1970)** $e_n = 2$ holds for $n = 3$.

- **(Post 1975 and Horak-Kim 2018)**: $e_n = \frac{\sqrt{2}}{2} n - \frac{3\sqrt{2}-2}{4}$ holds for $n \geq 6$ and $e_n = n - 1$ holds for $3 \leq n \leq 5$.


- **(Horak 2009)** $e_n = 2$ holds for $n = 5$. 
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- (Golomb-Welch 1970) $e_n = 2$ holds for $n = 3$.
- (Post 1975 and Horak-Kim 2018): $e_n = \frac{\sqrt{2}}{2} n - \frac{3\sqrt{2}-2}{4}$ holds for $n \geq 6$ and $e_n = n - 1$ holds for $3 \leq n \leq 5$.
- (Spacapan 2007) $e_n = 2$ holds for $n = 4$.
- (Horak 2009) $e_n = 2$ holds for $n = 5$.

Survey of papers on the Golomb-Welch conjecture:

The Golomb-Welch conjecture

Some results towards the Golomb-Welch conjecture (Fixed radius):

- (Horak-Grossek 2010) There are no linear 2-perfect Lee codes in $\mathbb{Z}^n$ for $n \leq 12$.

- (Kim 2017) There are no 2-perfect Lee codes in $\mathbb{Z}^n$ if $\#B^n(2) = 2n^2 + 2n + 1$ is a prime number satisfying certain conditions.

- (Qureshi-Campello-Costa 2018) There are no linear 2-perfect Lee codes in $\mathbb{Z}^n$ for infinitely many values of $n$.

- (Zhang-Ge 2017) For $e = 3$ and $e = 4$ there are no linear $e$-perfect Lee codes in $\mathbb{Z}^n$ for (possibly infinitely) many values of $n$. 
The Golomb-Welch conjecture

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Our main result:

If $e$ contains a digit 1 in its base-3 representation which is not in the unit place (e.g. $e = 3, 4$), there are no linear $e$-perfect Lee codes in $\mathbb{Z}^n$ for infinitely many values of $n$. 
1 Notations and definitions

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Notation

- \( \text{LPL}(n, e) = \{ C \subseteq \mathbb{Z}^n : C \text{ is a linear } e\text{-perfect Lee code} \} \).
- \( B^n(e) = \{ x \in \mathbb{Z}^n : d(x, 0) \leq e \} \).
- \( k(n, e) = \#B^n(e) = \sum_{i=0}^{\min\{n,e\}} 2^i \binom{n}{i} \binom{e}{i} \)
- \( p(n, e) = \sum_{i=1}^e 2^i \sum_{j=1}^{r-i+1} j^2 \binom{e-j}{i-1} \binom{n-1}{i-1} = \sum_{i=0}^e 2i^2 k(n-1, e-i) \)
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Criterion of non-existence of linear perfect Lee codes

Theorem (Horak-AlBdaiwi 2012). $LPL(n, e) \neq \emptyset$ if and only if there is an abelian group $G$ and a homomorphism $\phi : \mathbb{Z}^n \to G$ such that $\phi|_{B^n(e)} : B^n(e) \to G$ is a bijection.
**Notation**

- \( \text{LPL}(n, e) = \{ C \subseteq \mathbb{Z}^n : C \text{ is a linear } e\text{-perfect Lee code} \} \)
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- \( k(n, e) = \# B^n(e) \) \( = \sum_{i=0}^{\min\{n,e\}} 2^i \binom{n}{i} \binom{e}{i} \)
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**Criterion of non-existence of linear perfect Lee codes**

Theorem (Horak-AlBdaiwi 2012). \( \text{LPL}(n, e) \neq \emptyset \) if and only if there is an abelian group \( G \) and a homomorphism \( \phi : \mathbb{Z}^n \rightarrow G \) such that \( \phi|_{B^n(e)} : B^n(e) \rightarrow G \) is a bijection.

**Remark**

When \( k(n, e) \) is square-free, we can take \( G = \mathbb{Z}_{k(n,e)} \).
**Definition**

- A pair of positive integer \((n, e)\) satisfies the Zhang-Ge condition if it is a solution of

\[
\begin{cases}
  k(n, e) \equiv 3 \text{ or } 6 \pmod{9}, \\
  p(n, e) \equiv 0 \pmod{3}
\end{cases}
\]

- The Zhang-Ge set of \(e \geq 1\) is the set

\[
ZG(e) = \{ n \geq 1 : (n, e) \text{ satisfies the Zhang-Ge condition} \}.
\]

**Theorem (Zhang-Ge 2017)**

If \(n \in ZG(e)\) and \(k(n, e)\) is square-free, then \(\text{LPL}(n, e) = \emptyset\).

**Corollary (Zhang-Ge 2017)**

If \(k(n, 3)\) is squarefree and \(n \equiv 12 \text{ or } 21 \pmod{27}\) then \(\text{LPL}(n, 3) = \emptyset\). If \(k(n, 4)\) is squarefree and \(n \equiv 3, 5, 21 \text{ or } 23 \pmod{27}\) then \(\text{LPL}(n, 4) = \emptyset\).
Lemma

If $G$ is an abelian group with order $|G| = mp$ with $p$ prime and $p 
mid m$ then there is an epimorphism $\psi : G \to \mathbb{Z}_p$. In particular $\psi$ is an $m$-to-1 map.

Zhang and Ge use the criterion of Horak and AlBdaiwi in the proof of their theorem. Composing the homomorphism given by this criterion with a suitable epimorphism (using the previous lemma) we can skip the squarefree condition in the Zhang-Ge Theorem:

Theorem

If $n \in ZG(e)$, then $LPL(n, e) = \emptyset$.

Corollary

If $n \equiv 12$ or $21 \pmod{27}$ then $LPL(n, 3) = \emptyset$.

If $n \equiv 3, 5, 21$ or $23 \pmod{27}$ then $LPL(n, 4) = \emptyset$. 

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Non-existence of linear $e$-perfect Lee codes

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Question

- For which values of $e \geq 1$ is $ZG(e) \neq \emptyset$?
- For which values of $e \geq 1$ is $ZG(e)$ an infinite set?
Question

- For which values of $e \geq 1$ is $ZG(e) \neq \emptyset$?
- For which values of $e \geq 1$ is $ZG(e)$ an infinite set?

For which values of $e \geq 1$ there is an integer $n \geq 1$ such that $k(n, e) \equiv 3, 6 \pmod{9}$?
Non-existence of linear $e$-perfect Lee codes depending on the base-3 representation of $e$. 

| # | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
Theorem

There is an integer \( n \geq 1 \) such that \( k(n, e) \equiv 3, 6 \pmod{9} \) if and only if \( e \) contains a digit 1 in its base-3 representation.

Definition

For \( n \geq 1 \) we consider its base-3 representation \( n = \sum_{i=0}^{h-1} n_i 3^i \) with each \( n_i \in \{0, 1, 2\} \). The function \( \delta_3 : \mathbb{Z}^+ \to \mathbb{N} \cup \{\infty\} \) is given by

\[
\delta_3(n) = \begin{cases} 
\max\{i : n_i = 1\} & \text{if } n_i = 1 \text{ for some } i \geq 0, \\
\infty & \text{if } n_i \neq 1 \text{ for all } i \geq 0.
\end{cases}
\]

That is, if the base-3 representation of \( n \) does not contain a digit 1 then \( \delta_3(n) = \infty \). Otherwise, \( \delta_3(n) \) is the position of the most significant digit 1 in the base-3 representation of \( n \). For example \( \delta_3(24) = \infty \) since the base-3 representation of 24 is \( 220_3 = 2 \cdot 3^2 + 2 \cdot 3^1 + 0 \cdot 3^0 \) and \( \delta_3(64) = 2 \) since \( 64 = 2101_3 = 2 \cdot 3^3 + 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 \).
Theorem

There is an integer $n \geq 1$ such that $k(n, e) \equiv 3, 6 \pmod{9}$ if and only if $e$ contains a digit 1 in its base-3 representation.

Some ingredients of the proof

- **Lemma 1.** If $n = n_t \ldots n_0(3)$ and $e = e_t \ldots e_0(3)$, then
  \[ k(n, e) \equiv \prod_{i=0}^{t} k(n_i, e_i) \pmod{3}. \]

- **Lemma 2.** If $n = 3^{m+1} + 3^m$, $m \geq 2$ then
  \[ k(n, e) \equiv 1 - 3\left(\begin{array}{c} e \\ 3^{m-1} \end{array}\right) + 3\left(\begin{array}{c} e \\ 2 \cdot 3^{m-1} \end{array}\right) - 4\left(\begin{array}{c} e \\ 3 \cdot 3^{m-1} \end{array}\right) + 6\left(\begin{array}{c} e \\ 6 \cdot 3^{m-1} \end{array}\right) - 4\left(\begin{array}{c} e \\ 9 \cdot 3^{m-1} \end{array}\right) + 3\left(\begin{array}{c} e \\ 10 \cdot 3^{m-1} \end{array}\right) - 3\left(\begin{array}{c} e \\ 11 \cdot 3^{m-1} \end{array}\right) + \left(\begin{array}{c} e \\ 12 \cdot 3^{m-1} \end{array}\right) \pmod{9}. \]

- **Lemma 3.** Let $e = e_t \ldots e_0(3)$. Then,
  \[ \left(\begin{array}{c} e \\ k \cdot 3^{m-1} \end{array}\right) \equiv \left(\begin{array}{c} e_{m+2} \cdot 3^4 + e_{m+1} \cdot 3^3 + e_m \cdot 3^2 + e_{m-1} \cdot 3 + e_{m-2} \\ 0 \cdot 3^4 + k_2 \cdot 3^3 + k_1 \cdot 3^2 + k_0 \cdot 3 + 0 \end{array}\right) \pmod{9}. \]

- **Lemma 4.** Let $n = 3^{m+1} + 3^m$, $m \geq 2$ and $\delta_3(e) = m + 1$. Then,
  \[ k(n, e) \equiv 3, 6 \pmod{9}. \]
Proposition

Let $n = 3^{m+1} + 3^m$, $m \geq 2$, $e \geq 1$ such that $\delta_3(e) = m + 1$. Then, $p(n, e) \equiv 0 \pmod{3}$.

Corollary

If $\delta_3(e) \geq 3$, then $ZG(e) \neq \emptyset$ (since $3^{\delta_3(e)} + 3^{\delta_3(e)-1} \in ZG(e)$).

Proposition

If $\delta_3(e) \in \{1, 2\}$ then the Zhang-Ge set $ZG(e)$ has infinitely many elements.

The proof consist in proving the following facts for $a, b$ with $\delta_3(b) = \infty$.

i) If $a \in \{3, 5\}$ and $e = a + 3^2 b$, then $12 \in ZG(e)$.

ii) If $e = 4 + 3^2 b$, then $3 \in ZG(e)$.

iii) If $9 \leq a < 18$ and $e = a + 3^3 b$, then $12 \in ZG(e)$.

Proposition

If $\delta_3(e) = 0$ then $p(n, e) \not\equiv 0 \pmod{3}$. 
**Theorem**

Let $E = \{e \geq 1 : 1 \leq \delta_3(e) < \infty\}$ and $e \in E$. Then, there are no linear $e$-perfect Lee codes in $\mathbb{Z}^n$ for infinitely many values of $n$.

**Obs.** $E$ has density one as subset of the positive integers

Let $N \geq 1$ and $h \geq 1$ such that $3^{h-1} \leq N < 3^h$. We have

$$1 \geq \frac{\#\{e \leq N : 1 \leq \delta_3(e) < \infty\}}{N} = 1 - \frac{\#\{e \leq N : \delta_3(e) = 0\}}{N} - \frac{\#\{e \leq N : \delta_3(e) = \infty\}}{N} \geq 1 - \frac{\#\{e < 3^h : \delta_3(e) = 0\}}{3^{h-1}} - \frac{\#\{e < 3^h : \delta_3(e) = \infty\}}{3^{h-1}} = 1 - \frac{2^{h-1}}{3^{h-1}} - \frac{2^h}{3^{h-1}}$$

Since $h = \lfloor \log_3(N) \rfloor + 1 \to \infty$ when $N \to \infty$, from the inequalities above, $\text{dens}(E) = 1$. 

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Thank you!
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