Covering problems in hierarchical poset spaces over finite rings

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INMA - UFMS and UEMS - Ponta Porã
July - 2018
The famous Football pool problem
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Short-covering problem in hierarchical poset space
Predict the outcome of the 6 matches below:

1. Djurgaarden v Mariupol
   - HOME
   - DRAW
   - AWAY

2. Jagiellonia Bialystok v Rio Ave
   - HOME
   - DRAW
   - AWAY

3. Dinamo Brest v Atromitos
   - HOME
   - DRAW
   - AWAY

4. Lask Linz v Lillestrom
   - HOME
   - DRAW
   - AWAY

5. Aberdeen v Burnley
   - HOME
   - DRAW
   - AWAY

6. The Strongest v Wilstermann
   - HOME
   - DRAW
   - AWAY
“Which is the minimum number of bets necessary to guarantee n-1 correct results in n matches?”
Which is the minimum number of words in a code with the property that all words in the space $F^n_3$ are within Hamming distance 1 from some codeword.
Given integers $n \geq 1$, $q \geq 2$ and $R \geq 0$, an alphabet $A$ with $|A| = q$

$(A^n, d)$: the set of $n$-tuples with entries in $A$ endowed with a metric $d$.

$B(c, R) = \{x \in A^n : d(x, c) \leq R\}$: the ball with center $c \in A^n$ and radius $R$.

**Definition**

A subset $C$ of $A^n$ is a q-ary $R$-covering of $A^n$ if

$$\bigcup_{c \in C} B(c, R) = A^n.$$
The Covering Problem

- $K_q(n, R)$: the minimal size of a $q$-ary $R$-covering of length $n$. 
The Covering Problem

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Is to determine $K_q(n, R)$
... for hamming distance and finite fields.

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- **60’s:** the problem received a lot of contributions and the problem was introduced in the coding theory context in the 60’s as covering codes with radius 1.
- **80’s:** initially investigated for arbitrary $R$ by Carnielli.
- **Nowadays:** Still an **open problem**.
**Poset** $\mathcal{P}$: Partially ordered set on $\{1, 2, ..., n\}$.

**Order Ideal**: $I \subseteq \mathcal{P}$ is an *ideal* of $\mathcal{P}$ if $a \in I$, $b \in \mathcal{P}$ and $b \preceq a$ then $b \in I$.

**the ideal generated by** $A$: denote by $\langle A \rangle$ the smallest ideal containing $A$. 

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\[ P = ([8], \leq) \]
\[(\{8\}) = \{1, 2, 3, 6, 8\}\]
Rank of $j \in \mathcal{P}$: $l(j) = \max \{|C| : C \subset \langle j \rangle \text{ and } C \text{ is a chain} \}$

$l(8) = 3$  \hspace{1cm}  $l(7) = 2$
The $k$-th level of $P$. $H_k = \{i \in X; l(i) = k\}$, $H_1 = \{1, 2, 3, 4\}$, $H_2 = \{5, 6, 7\}$ and $H_3 = \{8\}$
Examples of Posets

Anti-Chain Poset

Figure: A ([7], ≤) anti-chain poset.
Examples of Posets

Chain Poset

Figure: A ([5], \leq) chain poset.
Examples of Posets

NRT Posets

Figure: A ([9], \leq) NRT poset
Hierarchical Poset

Figure: A (9; 4, 2, 3) hierarchical poset.
\( \mathbb{X}_q^n \): set of \( n \)-tuples with entries in a finite ring with \( q \) elements.

Support of a vector: \( \text{supp}(x) := \{ i \in P : x_i \neq 0 \} \).

\( P \)-weight \( (\omega_P) \): \( \omega_P(x) := |\langle \text{supp}(x) \rangle| \).

\( P \)-distance:
\[
d_P(x, y) = \omega_P(x - y).
\]

*Poset space*: \( (\mathbb{X}_q^n, d_P) \).
Let $\mathcal{A}_n$ be an antichain on $[n]$. The metric $d_{\mathcal{A}_n}$ is the classical Hamming distance of coding theory.

In 2008, Nakaoka and dos Santos introduced short-covering problem in Hamming spaces over finite rings:

Given a integer $R$, which is the minimum number of words in a code with the property that all words in the space $\mathbb{X}_q^n$ are within Hamming distance $R$ from a multiple of some codeword.
Let $C_n$ be a chain poset. In 2010, Yildiz et al. solved the covering and short covering problems on this poset space over finite rings.
Let \( n = mr \) and let \( \mathbb{Z}[n] \) be a disjoint union \( m \) of chains of length \( r \). The arising metric space is called the \( NRT \) space. In 2015, Castoldi and Carmelo explore the covering problem in NRT spaces:

The Covering Problem in Rosenbloom-Tsfasman Spaces

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Submitted: Jan 16, 2015; Accepted: Aug 14, 2015; Published: Aug 28, 2015
Mathematics Subject Classifications: 94B65, 06A06, 94B25
From now, we will use $\mathbb{H}\left[ n(n_1, n_2, ..., n_h) \right]$ to denote a hierarchical poset with $h$ levels.

A poset space defined by a hierarchical poset is called \textit{hierarchical poset space}.

$K_q^\mathbb{H}(n(n_1, n_2, ..., n_h); R)$: minimum size of a $R$-covering code in the hierarchical poset space.
Denote by $V^H(n(n_1, n_2, \ldots, n_h), R)$ the size (volume) of a ball of radius $R$ in the hierarchical poset space $(\mathbb{X}_q^n, d^H)$. It is easy to see that

**Theorem (Ball Covering Bound)**

$$K^H_q(n(n_1, n_2, \ldots, n_h); R) \geq \frac{q^n}{V^H(n(n_1, n_2, \ldots, n_h), R)}.$$
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**Theorem**

$$K^H_q(n(n_1, n_2, \ldots, n_h); R) \leq q^{n-R}.$$
A minimal 1-covering code of \((X^3_2, H[3(2, 1)])\) is

\[ C = \{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\} \]

So, \(K^H_2(3(2, 1); 1) = 4\).
Let $\delta_i = \sum_{j=1}^{i} n_j$ and for a given integer $R > 0$ set an integer $0 < l \leq h$ such that $\delta_{l-1} < R \leq \delta_l$ and $r = R - \delta_{l-1}$, where $\delta_0 = 0$. Note that, $1 \leq r_i \leq n_i$. 
Let $\delta_i = \sum_{j=1}^{i} n_j$ and for a given integer $R > 0$ set an integer $0 < l \leq h$ such that $\delta_{l-1} < R \leq \delta_l$ and $r = R - \delta_{l-1}$, where $\delta_0 = 0$. Note that, $1 \leq r_i \leq n_i$.

**Theorem**

$$K_q^{\mathbb{H}}(n(n_1, n_2, \ldots, n_h); R) = K_q(n_l, r)q^{n-\delta_l}.$$
Let \(\mathcal{P}\) and \(\mathcal{Q}\) posets on \([n]\) such that \(\mathcal{Q}\) is a refinement of \(\mathcal{P}\), that is, \(\mathcal{P} \subset \mathcal{Q}\). In this case, for \(x, y \in X_q^n\), is easy to see that \(d_{\mathcal{P}}(x, y) \leq d_{\mathcal{Q}}(x, y)\).
Let $\mathcal{P}$ and $\mathcal{Q}$ posets on $[n]$ such that $\mathcal{Q}$ is a refinement of $\mathcal{P}$, that is, $\mathcal{P} \subset \mathcal{Q}$. In this case, for $x, y \in \mathbb{X}_q^n$, is easy to see that $d_\mathcal{P}(x, y) \leq d_\mathcal{Q}(x, y)$.

**Theorem**

$$K_q^\mathcal{P}(n; R) \leq K_q^\mathcal{Q}(n; R).$$

Since, $A_n \subset \mathcal{P} \subset C_n$ for all poset $\mathcal{P}$ on $[n]$, we can derive the trivial upper bound for all poset metric space.

**Corollary**

$$K_q^\mathcal{Q}(n; R) = K_q^{A_n}(n; R) \leq K_q^{\mathcal{P}}(n; R) \leq K_q^{C_n}(n; R) = q^n - R.$$
Let $\mathcal{P}$ and $\mathcal{Q}$ posets on $[n]$ such that $\mathcal{Q}$ is a refinement of $\mathcal{P}$, that is, $\mathcal{P} \subseteq \mathcal{Q}$. In this case, for $x, y \in \mathbb{X}_q^n$, is easy to see that $d_\mathcal{P}(x, y) \leq d_\mathcal{Q}(x, y)$.

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**Theorem**

$$K_q^\mathcal{P} (n; R) \leq K_q^\mathcal{Q} (n; R).$$

**Corollary**

$$K_q(n; R) \overset{(1)}{=} K_q^{A_n} (n; R) \leq K_q^\mathcal{P} (n; R) \leq K_q^{C_n} (n; R) \overset{(2)}{=} q^{n-R}.$$
Short-covering in hierarchical poset spaces

- Ambient space: $\mathbb{X}_q^n$;
- metric: hierarchical
- Extended ball: $E(c, R) = \bigcup_{\alpha \in \mathbb{X}_q} B(\alpha c, R)$

**Definition**

*Short-covering* $H \subset \mathbb{X}_q^n$ is an *R-short covering* of a metric space $(\mathbb{X}_q^n, d)$ if

$$\bigcup_{h \in H} E(h, R) = \mathbb{X}_q^n.$$
On the hierarchical poset space $(X^3_3, \mathcal{H}[3(2, 1)])$ an 1-short covering code is given by

$$C = \{(0, 0, 1), (1, 1, 0), (1, 1, 1), (1, 1, 2)\}.$$ 

One can easily check that $C$ is minimal, so $C^d_\mathcal{H}(X^3_3, 3, 1) = 4$. 

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Short-covering in hierarchical poset spaces

- $D(X_q)^*$: zero-divisors of $X_q$, except the zero.
- $U(X_q)$: the unity elements of $X_q$.

**Theorem**

Let $\mathbb{H} = \mathbb{H}[n(n_1, n_2, \ldots, n_h)]$ be the hierarchical poset over $[n]$, with level distribution $n_1, n_2, \ldots, n_h$. For $i = 1, \ldots, h$, denote $\delta_i = \sum_{j=1}^{i} n_j$ and for a given integer $R > 0$ such that $\delta_{l-1} < R \leq \delta_l$, holds

(i) If $R = \delta_l$, then

$$C^\mathbb{H}(X_q, n, R) = C(X_q, n - R, 0).$$

(ii) If $\delta_{l-1} < R < \delta_l$, then

$$C^\mathbb{H}(X_q, n, R) \leq C(X_q, n_l, r) \left(q^{n-\delta_l} + s(s - 1)|U(X_q)|C(X_q, n - \delta_l, 0)\right) + C(X_q, n - \delta_l, 0),$$

where $r = R - \delta_{l-1}$ and $s$ is the number of orbits of $D(X_q)^*$.
Thank you!